Math 250A Lecture 7 Notes

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1 Free Groups

1.1 Free abelian groups

Definition 1.1. The *free abelian group* on n generators $g_1, g_2, \dots, g_n$ is the group of elements of the form $\sum_{i=1}^{n} n_i g_i$ with $n_i \in \mathbb{Z}$.

There exists a unique isomorphism $\mathbb{Z}^n \cong G$ taking $e_i \mapsto g_i$. In categorical terms, the free abelian group is the coproduct of n copies of Z.

$$\begin{tikzcd}
 & & X \arrow[d, dashed] \arrow[ddll] \arrow[ddl] \arrow[ddr] \arrow[ddrr] & & \\
 & & G \arrow[dll, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[dr, "\pi_3"'] \arrow[drr, "\pi_4"] & & \\
\mathbb{Z} & \mathbb{Z} & & \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

Call n, the exponent of $\mathbb{Z}$, the rank of the free abelian group. Is the rank determined by $G = \mathbb{Z}^n$? Yes, because the number of homomorphisms from $\mathbb{Z}^n \to \mathbb{Z}/2\mathbb{Z}$ is 2^n.

Proposition 1.1. *Any subgroup of $\mathbb{Z}^n$ is free of rank $\leq n$.*

Proof. Recall the proof that finitely generated abelian groups are products of cyclic groups. We showed that if A is a subgroup of $\mathbb{Z}^n$, we can find generators $g_1, \dots, g_n$ of $\mathbb{Z}^n$. So A is generated by $n_1 g_1, n_2 g_2, \dots$ for some n_i, making A free of rank $\leq n$. $\square$

1.2 The free group on $g_1, \dots, g_n$

1.2.1 Construction of the free group

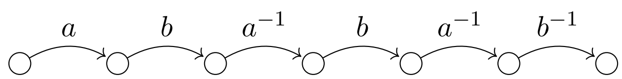
Take all words in symbols $g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1}$, including the empty word. For example 1, g_1, $g_1 g_2$, $g_1 g_2 g_1^{-1} g_2$, etc. These have an associative product, which is just concatenation of words. However, aa^{-1} is not the identity, so we still have some work to do. Take the smallest equivalence relation such that $g_1g_1^{-1} \equiv 1$, $g_2g_2^{-1} \equiv 1$,$g_3g_3^{-1} \equiv 1$, ... and such that if $a \equiv b$, then $ac \equiv bc$ and $ca \equiv cb$. This second condition ensures that the product is well-defined.

Definition 1.2. The *free group* on generators $g_1, \ldots, g_n$ is the group of equivalence classes of words in symbols $g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_n, g_n^{-1}$ under this equivalence relation, with the group operation being concatenation of words.

What does the free group look like? It can be identified with "reduced" words in $g_1, g_1^{-1}, \ldots$, where reduced means that we cancel out $g_1g_1^{-1}$, $g_2g_2{-}1$, etc.

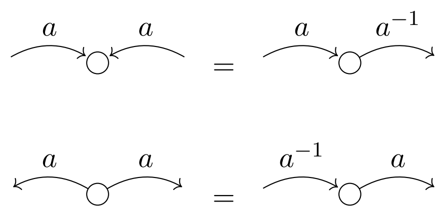
If A and B are different reduced words, are they different in the free group? Yes. It is sufficient to show that AB^{-1} is empty. Then it is sufficient to show that if A is a reduced word other than 1, the empty word, then $A \not\equiv 1$. We show that if $A \neq 1$ is a reduced word, we can find a finite group G and a map from the free group to G so that the image of A under this map is nontrivial. This is the statement that free groups are residually finite (nontrivial elements can be detected by finite groups).

Our finite group will be S_n for n = 1+length of the word A. We will illustrate the argument with an example. Let $A = b^{-1}a^{-1}ba^{-1}ba$. Draw the following graph on n vertices:



Map a to an element of S_{11} that respects the arrows on the graph; here, we must send a to a permutation σ_a with $\sigma_a(1) = 2$, $\sigma_a^{-1}(3) = 4$, and $\sigma_a^{-1}(5) = 6$, and we must send b to a permutation σ_b with $\sigma_b(2) = 3$, $\sigma_b(4) = 5$, and $\sigma_b^{-1}(6) = 7$. The constraints on a^{-1} become constants on a by noting that $\sigma_a^{-1}(x) = y \iff \sigma_a(y) = x$; the same holds for b^{-1}. Then A gets mapped to the permutation $\sigma_b^{-1}\sigma_a^{-1}\sigma_b\sigma_a^{-1}\sigma_b\sigma_a$, and this permutation sends the leftmost vertex, representing the element 1, to the rightmost vertex, representing the element $n = 7$.

There are two cases we need to watch out for. The first case is when we have two arrows labeled (without loss of generality) a going into the same vertex. The second case is when we have two arrows labeled a leaving the same vertex. But these are the graphs



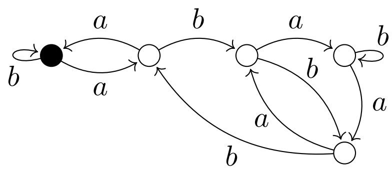
We cannot have aa^{-1} or $a^{-1}a$, lest these be reduced to the empty word. So our construction holds, and we are done.

For free abelian groups, a and b are isometries of the euclidean plane. For free groups, a and b are isometries of the hyperbolic plane.

1.2.2 Subgroups of free groups

Subgroups of free groups are free, but may have larger rank.

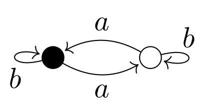
Look at the set of left cosets of G in F, and F acts on the left cosets. This gives the action of the generators $g_1, g_2, \ldots$ on the cosets. Pick one point, and call it the base point. This action determines a subgroup of index n of things fixing the base point. The number of subgroups of index n of F are in bijection with the connected graphs on n points with G-colored cycles.



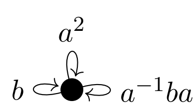
Free groups are the same as fundamental groups of connected graphs with a designated base point. The fundamental group is the set of homotopy classes of loops from the base point to itself, where loops follow the edges of the graph. Here, homotopy classes mean that two paths are equivalent if the difference between the two are just edges traversed that were immediately retraced backwards. The inverse of a path is the path traversed backwards.

The fundamental group of the graph containing 1 point with n loops to itself is the free group on n generators. Conversely, the fundamental group of any connected graph is a free group. Why? Pick an edge with distinct vertices; then we can contract the two points into one without changing the fundamental group. We can repeat this until there is only 1 point left, and we can then identify the fundamental group of the graph with a free group.

Example 1.1. Let G be a subgroup of index 2 of the free group on a, b.

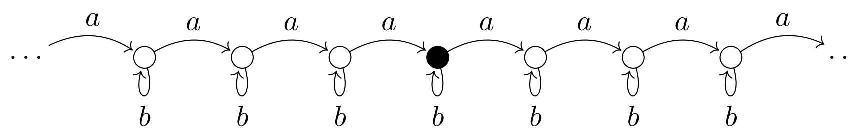


Contract the bottom edge, so we get a free group on 3 generators. What are these generators? Following the loops from the base point to itself, we get the generators b, a^2, and $a^{-1}ba$.



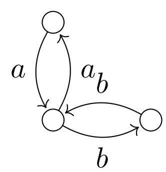
Not every subgroup of a finitely generated group is finitely generated.

Example 1.2. We will find a subgroup of the free group on 2 generators that is not finitely generated. Consider the following graph:

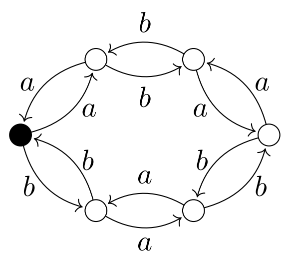


Contracting all the edges labeled a, we get 1 vertex with infinitely many loops going to itself. These loops are labeled with the generators $\{a^k b a^{-k} : k \in \mathbb{Z}\}$.

Example 1.3. Map the free group on two generators a, b to S_3 so a and b permute the vertices $\{1, 2, 3\}$ as follows:



S_3 is generated by a, b with the relations $a^2 = 1$, $b^2 = 1$, and $(ab)^3 = 1$. What is the kernel of this map? The kernel is a subgroup of index 6.



Contract along the inner edges to get the desired free group.

If G has index n in the free group on m generators, then the graph of F_m has Euler characteristic $1 - m$; the Euler characteristic of a graph is $|\text{vertices}| - |\text{edges}|$. The graph of G has n vertices and mn edges, so it has Euler characteristic $m - nm = n(1 - m)$. So the number of generators is $n(1 - m)$.